

Explosive synchronization transitions in complex neural networks

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It has been recently reported that explosive synchronization transitions can take place in networks of phase oscillators [Gómez-Gardeñes *et al.* Phys.Rev.Letts. 106, 128701 (2011)] and chaotic oscillators [Leyva *et al.* Phys.Rev.Letts. 108, 168702 (2012)]. Here, we investigate the effect of a microscopic correlation between the dynamics and the interacting topology of coupled FitzHugh-Nagumo oscillators on phase synchronization transition in Barabási-Albert (BA) scale-free networks and Erdős-Rényi (ER) random networks. We show that, if the width of distribution of natural frequencies of the oscillations is larger than a threshold value, a strong hysteresis loop arises in the synchronization diagram of BA networks due to the positive correlation between node degrees and natural frequencies of the oscillations, indicating the evidence of an explosive transition towards synchronization of relaxation oscillators system. In contrast to the results in BA networks, in more homogeneous ER networks the synchronization transition is always of continuous type regardless of the width of the frequency distribution. Moreover, we consider the effect of degree-mixing patterns on the nature of the synchronization transition, and find that the degree assortativity is unfavorable for the occurrence of such an explosive transition.

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I. INTRODUCTION

Synchronization is an emerging phenomenon of an ensemble of interacting dynamical units that is ubiquitous in nature, such as neurons, fireflies or cardiac pacemakers [1–3]. Inspired by the seminal works of small-world networks by Watts and Strogatz [4] and scale-free networks by Barabási and Albert [5], synchronization on complex networks has been widely studied [6–8]. These studies have revealed that the topology of a network has strong influences on the onset of synchronization [9–13], path towards synchronization [14], and the stability of the fully synchronized state [15–18]. However, the continuous nature of the synchronization phase transition is not affected by the topology of the underlying network, even in heterogeneous scale-free networks.

Recently, explosive transition in complex networks has received considerable attention since the discovery of an abrupt percolation transition in random networks [19, 20] and scale-free networks [21, 22], although some later studies claimed this transition is actually continuous but with unusual finite size behavior [23–25]. Along this line, very recently it was shown that explosive synchronization transitions can take place in scale-free networks of phase oscillators [26] and chaotic oscillators [27]. The mechanism responsible for such discontinuous synchronization transitions is the presence of a positive correlation between the heterogeneity of the connections and the natural frequencies of the oscillators. These studies

open new perspectives on the research of synchronization transitions of other dynamical systems. However, the research on this topic is still in its infancy and deserves more investigations.

In the present paper, we investigate the criticality of phase synchronization transition of coupled FitzHugh-Nagumo (FHN) oscillators in Barabási-Albert (BA) scale-free networks [5] and Erdős-Rényi (ER) random networks [28, 29]. It is shown that the positive correlation between degrees and natural frequencies of nodes can lead to a clear hysteresis loop in synchronization diagram of BA networks and thus signals the occurrence of an explosive synchronization transition in heterogeneous networks. And a sufficient wide distribution of natural frequencies of the oscillators is necessary to induce such an explosive transition in BA networks, while in more homogenous ER networks the synchronization transition is always continuous no matter how wide the distribution is. Furthermore, we consider the effect of degree-mixing patterns on the nature of the synchronization transition. We find that the degree assortativity is unfavorable for the occurrence of such an explosive transition.

II. MODEL

Let us consider a network of N coupled non-identical FHN oscillators, a representative model of excitable systems such as neurons, wherein the dynamics of the i th

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oscillator is described by the following equations [30],

$$\varepsilon \dot{x}_i = x_i - x_i^3 - y_i + C \sum_{j=1}^N A_{ij} (x_j - x_i), \quad (1)$$

$$\dot{y}_i = x_i + a_i + \xi_i(t). \quad (2)$$

The two dimensionless variables x and y are a voltage-like and a recovery-like variable, or in the terminology of physical chemistry and semiconductor physics, an activator and an inhibitor variable, respectively. The time scale ratio ε is much smaller than one (we here set $\varepsilon = 0.01$), implying x is the fast and y is the slow variable. The parameter a_i describes the excitability of the i th unit. If $|a_i| > 1$, the system is excitable, while $|a_i| < 1$ implies that the system is oscillatory. The element of adjacency matrix of the network takes $A_{ij} = 1$ if nodes i and j are connected, and $A_{ij} = 0$ otherwise. C is the coupling constant, and $\xi_i(t)$ is Gaussian noise that is independent for different units and satisfies $\langle \xi_i(t) \rangle = 0$ and $\langle \xi_i(t) \xi_j(t') \rangle = 2D\delta_{ij}\delta(t - t')$ with noise intensity D .

To establish a microscopic correlation between the dynamics and the topological properties of nodes, we assume that the natural frequency ω_i of the i th unit is an increasing function of its degree k_i , where $k_i = \sum_j A_{ij}$ is the number of nodes that are adjacent to the node. Since the oscillation frequency is a decreasing function of a_i , we thus consider a_i decreases linearly with k_i for simplicity, i.e.,

$$a_i = 0.99 - \delta \frac{k_i - k_{\min}}{k_{\max} - k_{\min}}, \quad (3)$$

where k_{\max} and k_{\min} are the maximum and minimum degree in the network, respectively. The factor δ determines the slope of the linear distribution. The larger δ is, the wider distribution of a_i has, or equivalently, a wider frequency distribution. Other parameters $N = 200$, $D = 0.005$, and the average degree $\langle k \rangle = 6$ are fixed in this paper unless otherwise specified.

To characterize synchronization behavior among the N oscillators, we first define the phase of the oscillator i as [31]

$$\phi_i(t) = 2\pi \frac{t - \tau_i^k}{\tau_i^k - \tau_i^{k+1}} + 2\pi k, \quad (4)$$

where τ_i^k is the time of the k th firing of the oscillator i , which is defined in simulation by threshold crossing of $x_i(t) = 1.0$. Thus, the degree of phase synchronization can be measured by calculating $r = \left\langle \left| \frac{1}{N} \sum_{j=1}^N e^{i\phi_j} \right| \right\rangle$, where the vertical bars denote the module and the angle brackets a temporal averaging. For completely unsynchronized motion $r \simeq 0$, while for fully synchronized state $r \simeq 1$.

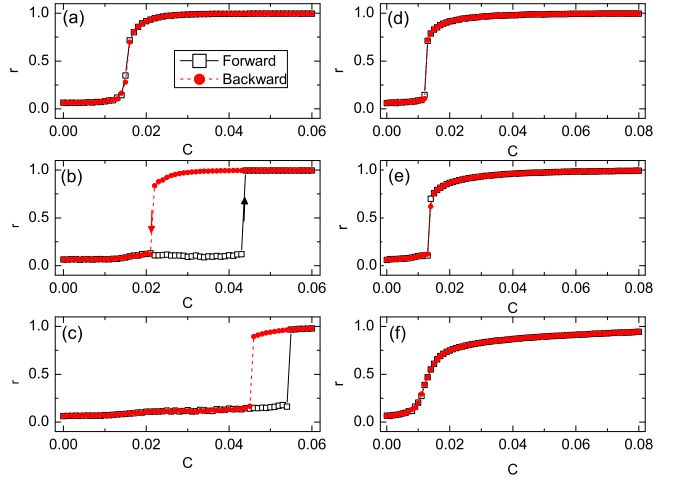


FIG. 1: (Color online) Phase synchronization degree r as a function of the coupling strength C for different δ in BA scale-free networks (left panels) and ER random networks (right panels). (a) and (d) for $\delta = 0.1$, (b) and (e) for $\delta = 0.3$, and (c) and (f) for $\delta = 0.9$. Squares (circles) in (a)-(f) mark the forward (backward) simulations, as C is increased (decreased) in steps of $C = 0.001$. Other parameters are $N = 200$, $\varepsilon = 0.01$, $D = 0.005$, and the average degree $\langle k \rangle = 6$.

III. RESULTS

The synchronization diagram is obtained by performing both forward and backward simulations. The former is done by calculating stationary value of r as varying C from 0 to 0.06 in steps of 0.001, and using the final configuration of the last simulation run as the initial condition of the next run, while the latter is performed by decreasing C from 0.06 to 0 with the same step. Fig.1 shows the results of r as a function of C for different values of δ in BA scale-free networks (left panels) and in ER random networks (right panels). For BA networks, one can find that the nature of synchronization transition drastically changes with δ . For $\delta = 0.1$ (Fig.1(a)), the results on the forward and backward simulations coincide, implying that the synchronization transition is continuous and of a second-order type. Increasing δ to $\delta = 0.3$ (Fig.1(b)), one can see that as C increases r abruptly jumps from $r \simeq 0$ to $r \simeq 1$ at $C = 0.044$, which show that a sharp transition takes place for the onset of synchronization. On the other hand, the curves corresponding to the backward simulations also shows a sharp transition from the synchronized state to the incoherent one at $C = 0.021$. The two sharp transitions occur at different values of C , leading to a strong hysteresis loop with respect to the dependence of r on C . Such a feature indicates that an explosive first-order synchronization transition arises in BA networks due to the positive correlation between degrees and natural frequencies. Further increasing δ to $\delta = 0.9$ (Fig.1(c)), the nature of first-order phase transition is still present, but the area of the hysteresis loop becomes smaller. While for ER random

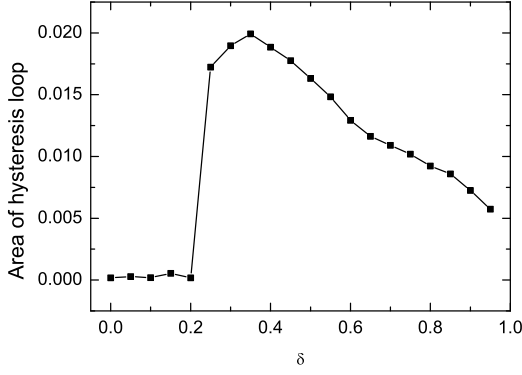


FIG. 2: Area of hysteresis loop in the synchronization diagram of BA scale-free networks as a function of δ . Other parameters are the same as Fig.1.

networks (Fig.1(d-f)), the forward and backward simulations always coincide regardless of δ , and thus such a correlation does not induce an explosive synchronization transition in ER networks. Therefore, a wide enough distribution of natural frequencies and degree heterogeneity are both necessary ingredients for the occurrence of such an explosive synchronization transition.

To show how the criticality of synchronization transition in BA networks changes with δ , we calculate the area of the hysteresis loop in synchronization diagram as a function of δ , as shown in Fig.2. One can see that this area equals to zero when $\delta \leq 0.2$, implying that the synchronization transition is of second-order. When δ is increased to $\delta = 0.25$, the value of this area drastically changes to a non-zero value. That is to say, the criticality of synchronization transition changes from a second-order type to a first-order one at between $\delta = 0.2$ and $\delta = 0.25$. With further increasing δ , this area show a nonmonotonic dependence on δ and a maximum area occurs at $\delta = 0.35$. This is consistent with the observation in Fig.1.

To further analyze the change of the order of the synchronization transition, we have calculated the average frequencies ω_k of nodes with degree k ($k \in [k_{\min}, k_{\max}]$) along the forward simulation, defined as

$$\omega_k = \frac{1}{N_k} \sum_{i|k_i=k} \frac{\phi_i(t+T) - \phi_i(t)}{2\pi T}, \quad (5)$$

where N_k is the number of nodes that have degree k , and $T \gg 1$. By monitoring the variation of ω_k with C one can clearly observe that how the full synchronization state is achieved. In Fig.3(a) we plot ω_k as a function of C in BA networks with $\delta = 0.3$. Before the synchronization happens, the average frequencies ω_k for nodes with small degrees and large degrees increase monotonically, while for those nodes with intermediate degree ω_k strongly fluctuate between two transition points. At $C = 0.044$, nodes with any degree class abruptly oscillate synchronously, which signals the explosive synchronization shown in Fig.1(b). For comparison, in Fig.3(b) and

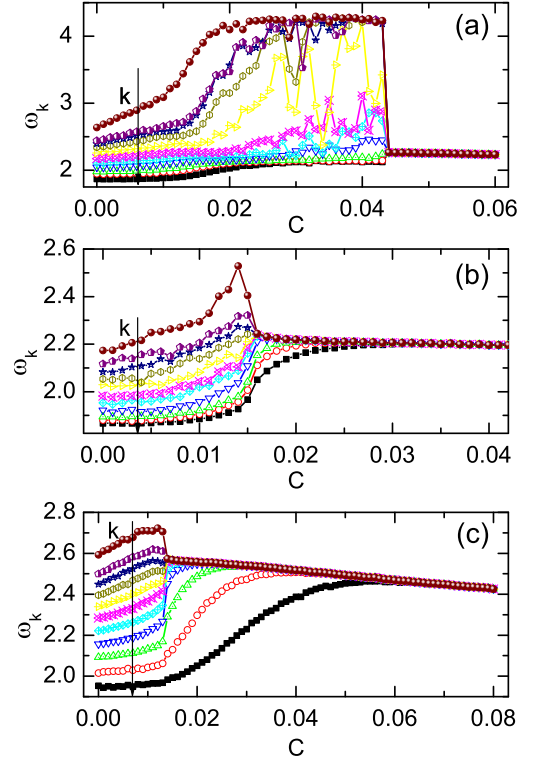


FIG. 3: (Color online) The average frequencies ω_k for different degrees k along the forward simulation. (a) BA networks: $\delta = 0.3$; (b) BA networks: $\delta = 0.1$; (c) ER networks: $\delta = 0.3$. The arrows in (a-c) indicate the decreasing order of degrees. Other parameters are the same as Fig.1.

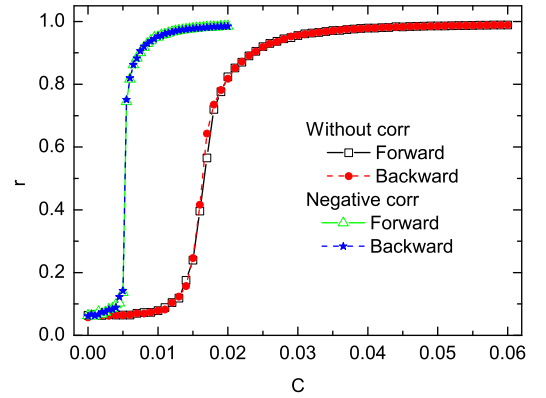


FIG. 4: (Color online) Synchronization diagram in BA scale-free networks without any correlation and with the negative correlation between degrees and natural frequencies. Other parameters are the same as Fig.1.

Fig.3(c) we also plot ω_k as a function of C in BA networks with $\delta = 0.1$ and in ER networks with $\delta = 0.3$, respectively. One can see that those nodes with large degree first become locked in frequencies while nodes with the small degree classes achieve full synchronization later on. This indicates that the synchronization transition is continuous.

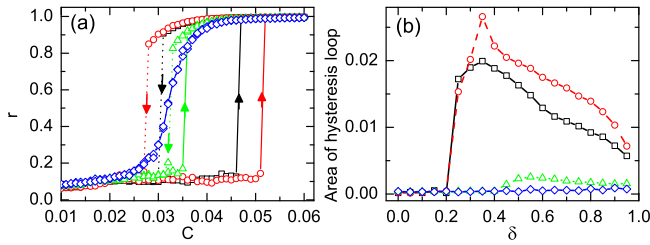


FIG. 5: (Color online) Effect of degree-degree correlations on the criticality of the synchronization transition. (a) Synchronization diagram at $\delta = 0.5$ for different degree-mixing coefficient r_k . (b) Area of hysteresis loop in synchronization diagram as a function of δ for different r_k . $r_k = 0.0$ (black squares), $r_k = -0.3$ (red circles), $r_k = 0.1$ (green triangles), and $r_k = 0.2$ (blue diamonds). Other parameters are the same as Fig.1.

Next we will illustrate whether the positive correlation between degrees and natural frequencies is responsible for the explosive synchronization. In Fig.4, we show the results of r as a function of C in BA networks, where the correlation is destroyed by randomly shuffle the values of a_i . One can see that the above-mentioned explosive synchronization transition disappears and the transition becomes a second-order type. Furthermore, we consider the case of negative correlation between degrees and natural frequencies. To the end, we set $a_i = 0.69 + 0.3(k_i - k_{\min})/(k_{\max} - k_{\min})$ while other parameters keep the same as Fig.1(a). The results are also shown in Fig.4 and the transition is continuous. Therefore, we can safely conclude that such an explosive synchronization transition arises due to the positive correlation between degrees and natural frequencies.

Lastly, we will consider the effect of degree-degree correlations on the criticality of the synchronization transition. It has been witnessed that many real networks display different degree-mixing patterns [32]. To measure the degree of the correlation, Newman defined a degree-mixing coefficient as [32]

$$r_k = \frac{M^{-1} \sum_i j_i k_i - [M^{-1} \sum_i \frac{1}{2}(j_i + k_i)]^2}{M^{-1} \sum_i \frac{1}{2}(j_i^2 + k_i^2) - [M^{-1} \sum_i \frac{1}{2}(j_i + k_i)]^2}, \quad (6)$$

where j_i and k_i are the degrees of nodes at the two ends of the i th link with $i = 1, \dots, M$ (M is the number of total links in the network). $r_k = 0$ indicates that there is no degree correlation, while $r_k > 0$ (< 0) indicates that a network is assortatively (disassortatively) mixed by degree. A previous study has revealed that degree-mixing pattern plays an important role on synchronization [33]. To generate different degree-mixing networks, we employ an algorithm proposed in [34]. At each elementary step, two links in a given network with four different nodes are randomly selected. To get an assortative network, the links are rewired in such a way that one link connects the two nodes with the smaller degrees and the other connects the two nodes with the larger degrees. Multiple

connections are forbidden in this process. Repeat this operation until an assortative network is generated without changing the node degrees of the original network. Similarly, a disassortative network can be produced with the rewiring operation in the mirror method. We start from BA scale-free networks with a neutrally degree-mixing pattern, and produce some groups of degree-mixing networks by performing the above algorithm. Fig.5(a) plots the synchronization diagram for different r_k at a fixed $\delta = 0.5$. One can see that for $r_k = -0.3$ the explosive synchronization transition persists. For $r_k = 0.1$ the discontinuous nature of the transition does not change, but the area of hysteresis loop become rather small. While for $r_k = 0.2$, the forward and backward simulations coincide and the synchronization transition becomes continuous. In Fig.5(b), we plot the area of hysteresis loop as a function of δ for different r_k . We find that the case for $r_k = -0.3$ is similar to that of $r_k = 0.0$. For $r_k = 0.1$ the area is larger than zero when $\delta > 0.045$ but its value is very small and has only the order of 10^{-3} . While for $r_k = 0.2$, the area is always zero regardless of δ , implying the nature of the synchronization transition has been changed essentially. Therefore, the degree-mixing patterns have a significant impact on the criticality of the synchronization transition. For a disassortative network, the explosive nature of the transition appears if δ is larger than a threshold value, while for a assortative network whose degree-mixing coefficient r_k is larger than 0.2, the explosive transition is absent and the transition becomes continuous. Since dissortativity implies nodes with larger degrees tend to connect to those nodes with smaller degrees, local star configurations are abundant in a disassortative network. Thus, it seems to suggest that the local star configurations are important for resulting in such an explosive phenomenon.

IV. SUMMARY

In summary, we have studied the effect of a microscopic correlation between degrees and natural frequencies of FHN oscillators on the criticality of synchronization transition in networks of BA and ER models. We find that, if the width of frequency distribution is larger than a critical value, the positive correlation degrees and oscillation frequencies can lead to the first-order synchronization transition in BA networks. While in more homogeneous ER network, such an explosive transition dose ont appear no matter how wide the distribution is. Therefore, the positive correlation between degrees and oscillation frequencies and their heterogeneities are both necessary conditions for such an explosive phenomenon. Moreover, we have shown the patterns of degree-degree correlations have a significant impact on the nature of the synchronization transition. In a disassortative network such an explosive phenomenon persists, while in an assortative network the transition becomes continuous type if the degree of assortative correlation is relatively

large. Our results generalize previous findings in phase [26] and chaotic oscillators [27] to relaxation oscillators with time-scale separation, and suggests that the mechanism for generating the discontinuous synchronization transition may be universal. In addition, an explosive transition may imply that the unsynchronized oscillation state can be metastable with respect to the full synchronized oscillation state near the synchronization transition point [35]. The dynamics of spontaneous synchronization from the metastable state to the stable full synchronized state without any change of system parameters may be

interesting and deserves further investigations.

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